Recall the setting:

- X complex torus, $V = T_0X \xrightarrow{\pi} X$ hom. of Lie gps., $\Lambda = \ker\pi \cong \mathbb{Z}^{2n}$, $V \cong \mathbb{C}^n$
- $\operatorname{Pic}X \Leftrightarrow \left\{ (H,\alpha) \;\middle|\; \begin{array}{l} H \text{ Hermitian on } V \\ \operatorname{Im}H(\Lambda\times\Lambda) \subseteq \mathbb{Z} \\ \alpha : \Lambda \to \{z\in\mathbb{C} : |z|=1\},\ \alpha(g+g') = \alpha(g).\alpha(g').e^{\pi i E(g,g')} \end{array} \right\}$

The line bundle $L(H,\alpha)$ associated to (H,α) is given by

$$L(H,\alpha) = \mathbb{C}\times V/_{(t,z)\sim(e_g(z)\cdot t,\, z+g)}$$ and $H^0(X, L(H,\alpha)) \leftrightarrow \Theta : V \to \mathbb{C}$ st $\Theta(z+g) = e_g(z).\Theta(z)$ (*)

- $L(H,\alpha)$ ample ($\Leftrightarrow$ by definition, $\exists n\in\mathbb{N}$ st L^n induces an embedding $X \overset{\phi_{L(H,\alpha)}}{\underset{\text{closed}}{\hookrightarrow}} \mathbb{P}^n$)
  $\Rightarrow$ H is positive def.

Today:

(1) Theorem 1: H positive definite $\Rightarrow \dim_{\mathbb{C}} H^0(X, L(H,\alpha)) = \sqrt{|\det E|}$

Corollary: H positive definite $\Rightarrow \phi_{L(H,\alpha)}$ exists as a rational map.

(2) Theorem 2: (Lefschetz) $L(H,\alpha)$ ample $\Leftrightarrow$ H pos. definite.

Proof of Thm 1:

Fourier transform: $f:\mathbb{C}\to\mathbb{C}$ holo. $f(z+1) = f(z) \Rightarrow c_n = \int f(z).\bar{e}^{2\pi i z n}$ and $f(z) = \sum c_n e^{2\pi i z n}$

We will use the multivariable version of this in this case $f(z+1) = f(z)$ becomes, $f(z+g) = f(z)$ for $g\in\Lambda' \cong \mathbb{Z}^n \subseteq \Lambda$ and $n \rightsquigarrow x \in \hat{\Lambda} = \operatorname{Hom}(\Lambda', \mathbb{Z})$.

Last time. We found $\Lambda' \subseteq \Lambda$ st

(a) $E(\Lambda'\times\Lambda') = 0$ (b) $\Lambda' \cong_{grp} \mathbb{Z}^n$ (c) $W := \Lambda'\otimes_{\mathbb{Z}}\mathbb{R} \Rightarrow W\oplus iW = V$ i.e. $\Lambda'\otimes_{\mathbb{Z}}\mathbb{C} = V$.

(d) $H|_{\Lambda'\times\Lambda'}$ is real symmetric let $B: V\times V\to\mathbb{C}$ the unique complex sym. extension.

(e) $H-B|_{W\times W} = 0$ using linearity in 1st variable $\Rightarrow H-B(z,w) = 0$ ($z\in V$, $w\in W$).

(f) $\alpha : \Lambda' \to \mathbb{C}_1^*$ is a Lie group homo. $\Rightarrow \alpha(g) = e^{2\pi i\lambda(g)}$ for $\lambda \in \operatorname{Hom}_{\mathbb{C}\text{-vs}}(V,\mathbb{C})$.
$\cong \operatorname{Hom}_{\mathbb{R}\text{-vs}}(W,\mathbb{R})$

The goal is to turn Θ into Λ' periodic using B and α.

Define $\Theta^*(z) = \Theta(z).e^{-1/2\pi B(z,z)}$. $\Theta(z+g) = \alpha(g).e^{\pi H(z,g)+\frac{1}{2}H(g,g)}\Theta(z)$.

$$e^{-1/2\pi B(z+g,z+g)} = e^{-1/2\pi B(z,z)+\pi B(z,g)-\frac{1}{2}\pi B(g,g)}$$

$\Rightarrow \Theta^*(z+g) = \alpha(g).e^{\pi(H-B)(z,g)+\frac{1}{2}(H-B)(g,g)}.\Theta^*(z)$ if $g\in\Lambda'$ then the exponential term is $=1$ and $\alpha(g) = e^{2\pi i\lambda(g)} \Rightarrow e^{-2\pi i\lambda(z)}.\Theta^*(z)$ is Λ' periodic.

$$\Rightarrow e^{-2\pi i\lambda(z)}\Theta^*(z) = \sum_{x\in\hat{\Lambda}} c_x\, e^{2\pi i x(z)} \Rightarrow \Theta^*(z) = \sum_{x\in\hat{\Lambda}} c_x.e^{2\pi i(x(z)+\lambda(z))} \quad (**)$$

(*) gives for c_x: let $g\in\hat{\Lambda} = \operatorname{Hom}_{\mathbb{Z}}(\Lambda',\mathbb{Z}) \subseteq \operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$ $\hat{g}(g') = E(g',g)$.

$\Rightarrow$

$$c_x = \alpha(g).e^{\pi\hat{g}(g)-2\pi i(x(g)+\lambda(g))}.c_{x-\hat{g}} \quad (\Delta) \text{ where } g\in\Lambda\subseteq\mathbb{C}.$$

This equation characterizes Θ as soon as (**) converges.

- $\exists$ mtx of E in the form $\left(\begin{array}{c|c} 0 & A \\ \hline -A^T & 0 \end{array}\right)$ then the map $\begin{array}{l}\Lambda \to \hat{\Lambda}' \\ g \mapsto \hat{g} = pr_{\Lambda''}(g).A.(-)\end{array}$

is actually A.

$|\det E| = |\det A|^2 \Rightarrow \sqrt{|\det E|} = |\det A| = |\overbrace{\operatorname{coker}A}^{M}|$, it rests to find a correspondance,
$\{\text{cosets of } M\} \underset{\mathbb{C}\text{-lin.}}{\Longleftrightarrow} \{\text{basis of } \Theta \text{ func.}\}$.

$(\Delta) \Rightarrow$ c_x on any representative of any coset of M determines the value of c_x on the whole coset.

Proof of Theorem 2:

Thm 1 $\Rightarrow \exists \Theta \neq 0 \in H^0(L(H,\alpha))$, consider $\Theta(z-a)\Theta(z-b)\Theta(z+a+b) = \Theta^{a,b}(z)$ for $a,b \in V$.

Claim: $\Theta^{a,b}$ is a Θ-function.

Proof: $\Theta^{a,b}(z+g) = \underbrace{\alpha_H(g)^3}_{\alpha_H^3 = \alpha_{3H}} . e^{\pi \underbrace{H(3z,g)}_{3H(z,g)} + \underbrace{3/2 H(g,g)}_{\frac{1}{2}3H(g,g)}} . \Theta^{a,b}(z)$ $\square$

Claim: $\forall z \; \exists a,b$ st $\Theta^{a,b}(z) \neq 0$.

$\Rightarrow L(H,\alpha)^3$ defines a morphism $X \xrightarrow{\xi} \mathbb{P}^n$ as there are non zero sections at each point

Replace $L(H,\alpha)$ with $L(H,\alpha)^3$.

(up to replacing L by L^3)

Need: ξ injective: Suppose $\xi(\pi(z_1)) = \xi(\pi(z_2))$ st $\pi(z_1) \neq \pi(z_2) \Rightarrow \exists \gamma \in \mathbb{C}^\times$ st $\Theta(z_1) = \gamma . \Theta(z_2)$ & $\sigma := z_2 - z_1 \in \Lambda$.

Point of the proof: Every Θ fun for Λ is also a Θ-function for $\Lambda \oplus \sigma\mathbb{Z}$.

$\Rightarrow \det(\Lambda \oplus \sigma\mathbb{Z}) < \det(\Lambda) \Rightarrow$ contradiction to thm 1.

$\Theta(z_2-a) . \Theta(z_2-b) . \Theta(z_2+a+b) = \gamma \Theta(z_1-a) . \Theta(z_1-b) . \Theta(z_1+a+b)$.

Take the log. derivative wrt a. Let $\omega = \log(\Theta(-))'$.

$\Rightarrow -\omega(z_2-a) + \omega(z_2+a+b) = -\omega(z_1-a) + \omega(z_1+a+b) \Rightarrow \omega(z_2+z) - \omega(z_2-z)$ is translation invariant $\Rightarrow$ it is constant

$\Rightarrow$ after integrating, $\Theta(z+z_2) = e^{\ell(z)}\Theta(z+z_1)$ with ℓ linear $\Rightarrow \Theta(z+\sigma) = \overbrace{A}^{\in \mathbb{C}^\times} . e^{\ell(z)}\Theta(z)$